

Computing multiway cut within the given excess over the largest minimum isolating cut

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Abstract

Let (G, T) be an instance of the (vertex) multiway cut problem where G is a graph and T is a set of terminals. For $t \in T$, a set of nonterminal vertices separating t from $T \setminus \{t\}$ is called an *isolating cut* of t . The largest among all the smallest isolating cuts is a natural lower bound for a multiway cut of (G, T) . Denote this lower bound by m and let k be an integer.

In this paper we propose an $O(kn^{k+3})$ algorithm that computes a multiway cut of (G, T) of size at most $m + k$ or reports that there is no such multiway cut. The core of the proposed algorithm is the following combinatorial result. Let G be a graph and let X, Y be two disjoint subsets of vertices of G . Let m be the smallest size of a vertex $X - Y$ separator. Then, for the given integer k , the number of *important* $X - Y$ separators [16] of size at most $m + k$ is at most $\sum_{i=0}^k \binom{n}{i}$.

1 Introduction

1.1. Results and motivation. Let (G, T) be a pair where G is a graph and T a subset of $V(G)$. Let us call the vertices of T the *terminals*. A *multiway cut* of (G, T) is a set S of non-terminal vertices such that in $G \setminus S$ no two terminals belong to the same connected component. The multiway cut problem MWC asks for the smallest multiway cut of (G, T) . For two terminals this problem can be solved by network flow techniques but becomes NP-hard for 3 terminals [8].

Let $t \in T$. An *isolating cut* of t [8] is a set $S \subseteq V(G) \setminus T$ separating t from the rest of terminals. Denote by $m(t)$ the size of the smallest isolating cut of t and let $m = \max_{t \in T} m(t)$. It is not hard to see that m is a *polynomially computable lower bound* on the size of the smallest multiway cut of (G, T) .

In this paper, we investigate computing a multiway cut with a bounded *excess* over m . In particular, the main result of this paper is an $O(kn^{k+3} + |T|n^3)$ time algorithm that checks whether (G, T) has a solution of size at most $(m + k)$ for the given integer k .

The main motivation of the above result comes from Parameterized Complexity [10]. The MWC problem is well-known to be Fixed-Parameter Tractable (FPT) parameterized by the solution size [16, 5]. Can we provide a better parameterization that addresses the case where the optimal solution is large? For some problems, positive answers have been obtained by introducing so called *parameterization above guaranteed value*, the idea first proposed in [15]. We apply this template to the MWC problem. In our case, the guaranteed value is m (because the solution size is always m or greater) and we study the parameterization of the MWC problem by the excess over m . The proposed result makes a progress in this study because it shows that the MWC problem with respect to the considered parameter is in XP and this makes meaningful the question as to *whether the MWC problem is FPT parameterized by the excess over the maximal size of a smallest isolating cut*. To the best of our knowledge this is the first result addressing the MWC problem parameterized above a guaranteed value.

The key ingredient in the proof of the above result is a combinatorial theorem bounding the number of *important* $X - Y$ separators [16] of excess at most k over the smallest one. Let X and Y be two disjoint subsets of $V(G)$. Let r be the size of a smallest $X - Y$ separator. It is known [16] that there is exactly one important $X - Y$ separator of size r . But how many are there important separators of size at most $r + k$ for the given integer k ? The best existing bound is 4^{r+k} [14, 18]. We prove that the number of such important separators is at most $\sum_{i=0}^k \binom{n}{i}$, which is much better than 4^{r+k} if r is large. To the best of our knowledge, this is the first upper bound on the number of important separators where the size of a separator is not in the exponent. This upper bound is obtained by observing that important separators have a number of nice structural properties that establish an injective function from the set of important separators of size at most $r + k$ to the family of subsets of vertices of size at most k .

1.2. Related work. The MWC problem is a natural generalization of the standard $s - t$ cut problem having applications related to resource allocation such as Multiprocessor Scheduling [20] and Medical Imaging [2, 1]. This problem has been shown NP-hard in [8] even for the case of three terminals. This gave rise to the investigation of methods of coping with NP-hardness for the MWC problem. In the direction of identifying polynomially solvable subclasses, the researchers mainly concentrated on planarity and tree-like structures (e.g. [9, 8, 13]). Approximation algorithms for this problem have been also actively investigated resulting in a row of improvements and generalizations (see e.g. [8, 3, 11]).

The notion of *isolating cut* (for the edge MWC problem) has been coined first in [8] in connection to the design of an approximation algorithm. In [11], the notion has been reformulated in terms of the vertex MWC problem in the way used in the present paper. However, in [11] it is pointed out that their algorithm is not based on this notion.

The parameterized version of the MWC problem was first considered in [16], the solution size being the parameter. An algorithm with a significantly improved runtime has been proposed in [5]. The key theorem behind this algorithm gave rise to first FPT algorithms for the Directed Feedback Vertex Set [6] and Min 2-CNF deletions problems [19], whose fixed-parameter tractabilities were long standing open questions. We believe this is an indication that the MWC problem is a very convenient framework for studying graph separation problems in the sense that it reveals some structural properties relevant to many other problems but not easily seen there.

The notion of *important separator* has been coined in [16]. It is explicitly used in [14] and [18] for resolving a number of challenging open problems. In fact, as pointed out in [14], [5, 6, 19] also implicitly use important separators. This shows that an important separator is an interesting and worth studying combinatorial concept.

Finally, the investigation of parameters above and below guaranteed values has been initiated in [15]. Currently, it is an active research area. An overview of it can be found in the introduction of [12].

1.3. Structure of the paper. Section 2 introduces the necessary background notions and their basic properties. Section 3 introduces the notion of *important witness*, a special case of important separator, having some nice properties. Using these properties, Section 4 shows that any non-smallest important separator is nothing else but a *compound witness*, a generalization of an important witness uniquely associated with a subset of vertices of size not greater than its excess. From this the desired upper bound on the number of important separators is derived and applied to the MWC problem.

2 Preliminaries

We employ a standard notation related to graphs. In particular, given a graph G , let $C \subseteq V(G)$. Then $G[C]$ denotes the subgraph of G induced by C and $G \setminus C \equiv G[V(G) \setminus C]$. For $v \in V(G)$, $G \setminus v \equiv G[V(G) \setminus \{v\}]$ and $N(v)$ is the set of neighbors of v in G . Also, $N(C) \equiv (\bigcup_{v \in C} N(v)) \setminus C$.

Let X and Y be two disjoint sets of vertices of the given graph G . A set $K \subseteq V(G) \setminus (X \cup Y)$ is an

$X - Y$ separator if in $G \setminus K$ there is no path from X to Y . Let A, B be two disjoint subsets of $V(G)$. We denote by $NR(G, A, B)$ the set of vertices that are not reachable from A in $G \setminus B$. Let K_1 and K_2 be two $X - Y$ separators. We say that $K_1 \geq K_2$ if $NR(G, Y, K_1) \supseteq NR(G, Y, K_2)$.

Proposition 1 *Let K_1 and K_2 be two minimal $X - Y$ separators. Then $K_1 \leq K_2$ if and only if $K_1 \setminus K_2 \subseteq NR(G, Y, K_2)$.*

Proof. Assume first that $K_1 \leq K_2$. Due to the minimality of K_1 , each $v \in K_1$ is adjacent to some vertex w of $NR(G, Y, K_1)$. Since $w \in NR(G, Y, K_2)$ by our assumption, $v \in NR(G, Y, K_2)$ whenever $v \in K_1 \setminus K_2$. For the opposite direction, any vertex of $NR(G, Y, K_1)$ can be connected to Y only through K_1 . Since in $G \setminus K_2$, all vertices of $K_1 \setminus K_2$ are disconnected from Y such connection is impossible. ■

Let K_1 and K_2 be two minimal $X - Y$ separators. Let $K_1^t = K_1 \cap NR(G, Y, K_2)$, $K_1^b = (K_1 \setminus K_1^t) \setminus (K_1 \cap K_2)$. Accordingly, let $K_2^t = K_2 \cap NR(G, Y, K_1)$ and $K_2^b = (K_2 \setminus K_2^t) \setminus (K_1 \cap K_2)$ (the superscripts 't' and 'b' correspond to the words 'top' and 'bottom'). We denote $K_1^t \cup K_2^t \cup (K_1 \cap K_2)$ and $K_1^b \cup K_2^b \cup (K_1 \cap K_2)$ by, respectively, $Top_{G,X,Y}(K_1, K_2)$ and $Bottom_{G,X,Y}(K_1, K_2)$, the subscripts may be omitted if they are clear from the context.

Proposition 2 *Let the notation be as in the previous paragraph. Then both $Top(K_1, K_2)$ and $Bottom(K_1, K_2)$ are $X - Y$ separators. Moreover, $Bottom(K_1, K_2) \geq K_1$ and $Bottom(K_1, K_2) \geq K_2$.*

Proof. Consider the set $N^* = NR(G, Y, K_1) \cup NR(G, Y, K_2)$. By definition of K_1 and K_2 this set includes X and does not contain any vertex of Y . What is the set of neighbors of this set, i.e. what is the set separating N^* from the rest of the graph? Clearly, it is a subset of $K_1 \cup K_2$ excluding those vertices that belong to $NR(G, Y, K_1) \cup NR(G, Y, K_2)$. In other words, it is a subset of $Bottom(K_1, K_2)$, and no vertex of $Bottom(K_1, K_2)$ belongs to N^* . It follows that $Bottom(K_1, K_2)$ is $X - Y$ separator, separating from Y a superset of $NR(G, Y, K_1)$ and of $NR(G, Y, K_2)$, i.e. $Bottom(K_1, K_2) \geq K_1$ and $Bottom(K_1, K_2) \geq K_2$ as required. ■

A minimal $X - Y$ separator K is called *important* if there is no $X - Y$ separator K' such that $K < K'$ and $|K| \geq |K'|$. This notion was first introduced in [16] in a slightly different form. In particular, let $R(G, X, K)$ be the set of vertices that belong to the same component in $G \setminus K$ with at least one vertex of X . In the definition of [16], the condition $K < K'$ is replaced by $R(G, X, K) \subset R(G, X, K')$. The following proposition shows that these conditions are equivalent thus implying the equivalence of definitions.

Proposition 3 *Let K and K' be two distinct $X - Y$ separators of G . Then $NR(G, Y, K) \subset NR(G, Y, K')$ if and only if $R(G, X, K) \subset R(G, X, K')$.*

Proof. It is not hard to see that since $K \neq K'$, $NR(G, Y, K) \neq NR(G, Y, K')$ and $R(G, X, K) \neq R(G, X, K')$. Indeed, if K is a minimal separator then K is the neighborhood of both $NR(G, Y, K)$ and $R(G, X, K)$, the same is, of course true for K' . But the same set cannot have two different neighborhoods. It follows that we can replace ' \subset ' by ' \subseteq ' in the statement of the observation. Assume that $NR(G, Y, K) \subseteq NR(G, Y, K')$ and let $v \in R(G, X, K)$. Then there is a $X - v$ path p all vertices of which belong to $R(G, X, K) \subseteq NR(G, Y, K) \subseteq NR(G, Y, K')$. It follows that v is reachable from X in $G \setminus K'$, i.e. $v \in R(G, X, K')$. Conversely, assume that $R(G, X, K) \subseteq R(G, X, K')$. Due to the minimality of K , each $v \in K$ is adjacent to a component C of $G \setminus K$ containing at least one vertex of X . Since all the vertices of C are preserved in $R(G, X, K')$, $v \in R(G, X, K') \subseteq NR(G, Y, K')$ whenever $v \in K \setminus K'$. The desired statement now follows from Proposition 1. ■

Corollary 1 *Let r be the size of a smallest $X - Y$ separator of G . Then there is exactly one important $X - Y$ separator K of size r . Moreover, $K^* > K$ for any other important separator K^* .*

Proof. Having in mind Proposition 3, the first statement is Lemma 3.3. of [16] and the second statement (in fact, both of them) are proven in the second and third paragraphs of the proof of Lemma 2.6. of [18]. ■

For the result proposed in this paper, we will need to compute the unique smallest important $X - Y$ separator. It is known to be polynomially computable, see, for example Lemma 3.2. of [16] for a more general polynomial computability statement. In the following lemma, we show that computing the smallest important $X - Y$ separator in fact takes the same time as computing an arbitrary smallest $X - Y$ separator.

Lemma 1 *The smallest important $X - Y$ separator can be computed in $O(n^3)$ by an algorithm that first computes in $O(n^3)$ a largest set of internally vertex disjoint $X - Y$ paths and then spends additional $O(n^2)$ time to computing the smallest important $X - Y$ separator.*

Proof. Let p_1, \dots, p_r be a largest set of internally vertex disjoint $X - Y$ paths that can be computed in $O(n^3)$ using standard network flow techniques (the computation takes at most $n + 1$ iterations of Ford-Fulkerson algorithm each taking $O(n^2)$, see, for example [7]). We are going to show how to compute the smallest important $X - Y$ separator having these paths computed. Assume that each p_i is of length r_i and enumerate its vertices $v_{i,1}, \dots, v_{i,r_i}$ in the order they occur in p_i being explored from X to Y . We may assume that for each p_i $v_{i,1}$ is the only vertex of X and v_{i,r_i} is the only vertex of Y otherwise we can just shorten these paths to obtain the desired effect. We can also assume that X and Y are singletons $\{x\}$ and $\{y\}$, respectively: for the purpose of the considered problem X and Y can be safely contracted into single vertices.

We use the concept of *torso* introduced in [17]. Recall that for $S \subseteq V(G)$, $torso(G, S)$ is the graph obtained from $G[S]$ by introducing new edges between those vertices v_1, v_2 of S that are connected by path all intermediate vertices of which lie outside S . Denote $V(p_1) \cup \dots \cup V(p_r)$ by V^* and consider the graph $torso(G, V^*)$. It follows from the combination of Proposition 2.5. in [17] and Proposition 1 that a set K is the smallest important separator of G if and only if it is the smallest important separator of G^* . Therefore the algorithm first constructs graph G^* and then solves the problem regarding G^* .

The algorithm consists of a number of iterations. On the i -th iteration the algorithm either computes a set S_i or returns the answer. The algorithm starts from setting $S_0 = \{y\}$. Assume that the algorithm is in the i -th iteration while it did not return the answer on the $i - 1$ -th iteration. For $1 \leq j \leq r$, let z_j be the largest index such that $v_{j,z_j} \notin S_{i-1}$ and let y_j be the smallest index such that v_{j,y_j} is adjacent to S_{j-1} . If for each j , $y_j = z_j$, the algorithm returns the set $\{v_{1,y_1}, \dots, v_{r,y_r}\}$. Otherwise, the algorithm obtains S_i by adding to S_{i-1} the vertices $v_{j,y_j+1}, \dots, v_{j,z_j}$ for each j such that $y_j \neq z_j$.

To analyze the algorithm, observe first that by construction $S_0 \subset S_1 \subset S_2 \dots$ and that for each S_i the subset of each $V(p_j)$ that belongs to S_i forms a suffix of p_j . It follows from the latter statement that each $G^*[S_i]$ is connected. Furthermore, observe that no S_i intersects with a smallest $X - Y$ separator. This is certainly true for S_0 . Assume the truth for S_{i-1} . If this is not the case for S_i then there is a vertex w of a smallest $X - Y$ separator K' that belong to the subpath of some p_j whose end vertices are v_{j,y_j+1} and v_{j,z_j} as defined above. It follows that K' does not contain any other vertex of p_j . Consequently, Y can be reached from X in $G^* \setminus K'$ by going along p_j from x to v_{j,y_j} and then jumping to S_{j-1} which is connected and disjoint with K' . This contradiction shows that correctness of the considered observation. It follows that each smallest $X - Y$ separator is in fact $X - S_i$ separator for all S_i generated during the run of the algorithm. Since S_i grows with the increase of i , the stopping condition is met after some $b+1 \leq n$ iterations (i.e. the last constructed set is S_b). It is not hard to observe that the returned set K is a smallest $X - Y$ separator. In fact it is also the desired important separator. Indeed, by the proven above the component S_b of Y in $G^* \setminus K$ is smallest possible in case we consider only smallest $X - Y$ separators. Consequently, $NR(G^*, Y, K) = V(G^*) \setminus (K \cup S_b)$ is largest possible. This finishes the correctness proof of the proposed algorithm.

For the runtime, note that G^* can be constructed in $O(n^2)$. The i -th iteration of the algorithm examines adjacency of S_{i-1} with the rest of the graph. But in fact we can consider only adjacency of $S_{i-1} \setminus S_{i-2}$

because the only vertices outside S_{i-1} adjacent to S_{i-2} are $v_{1,z_1}, \dots, v_{r,z_r}$ known by construction of S_{i-1} . It follows that the adjacency of each pair of vertices is examined a constant number of times and hence the algorithm takes time $O(n^2)$. ■

Definition 1 Let G be a graph and X, Y be two disjoint subsets of its vertices. We say that G is $X - Y$ normalized if $N(X)$ is the only smallest $X - Y$ separator.

Let K be a $X - Y$ separator. Denote by $Pr(G, X, Y, K)$ the graph obtained from $G \setminus (NR(G, Y, K) \setminus X)$ by making X adjacent to all the vertices of K . The graph $Pr(G, X, Y, K)$ has the following easily observable properties.

Proposition 4 1. Let $K_1 \geq K$ be an $X - Y$ separator. Then K_1 is a $X - Y$ separator of $Pr(G, X, Y, K)$. Moreover, if K_1 is a smallest $X - Y$ separator of G then K_1 remains a smallest $X - Y$ separator of $Pr(G, X, Y, K)$.

2. Let $K_2 \geq K$ be another $X - Y$ separator. Then $K_2 \geq K_1$ in G if and only if $K_2 \geq K_1$ in $Pr(G, X, Y, K)$. In particular, K_2 is an important $X - Y$ separator of G if and only if K_2 is an important $X - Y$ separator of $Pr(G, X, Y, K)$.

3. If K is an important $X - Y$ separator of G then $Pr(G, X, Y, K)$ is $X - Y$ normalized.

Proof. For part 1, consider an $X - Y$ path p in $Pr(G, X, Y, K)$. This path can be transformed into an $X - Y$ path of G , possibly by introducing vertices of $NR(G, Y, K)$. K_1 is disjoint with $NR(G, Y, K)$ by Proposition 1. On the other hand, K_1 intersects the transformed path. Consequently, K_1 intersects the initial path p . That is, K_1 is an $X - Y$ separator of $Pr(G, X, Y, K)$. Furthermore, since any $X - Y$ separator of $Pr(G, X, Y, K)$ is clearly an $X - Y$ separator of G , any smallest $X - Y$ separator of G is also a smallest separator of $Pr(G, X, Y, K)$.

For part 2, apply Proposition 1 and, arguing as in the previous paragraph, observe that K_2 separates $K_1 \setminus K_2$ in G if and only if the same happens in $Pr(G, X, Y, K)$. Finally, for part 3, observe that if K is not the only smallest separator of $Pr(G, X, Y, K)$ then K is not important in $Pr(G, X, Y, K)$ in contradiction to part 2. ■

3 Important witnesses

Definition 2 Let G be a graph, X, Y be two disjoint subsets of vertices, r be the smallest size of a $X - Y$ separator and K be an arbitrary $X - Y$ separator. We call $|K| - r$ the excess of K and denote it by $excess_{G,X,Y}(K)$, the subscripts may be omitted if clear from the context.

Definition 3 Let G be a $X - Y$ -normalized graph and let $S \subseteq N(X)$. We call the excess of a smallest $X - Y$ separator disjoint with S the cover excess of S and denote it by $CE_{G,X,Y}(S)$, the subscripts can be omitted if clear from the context. If S is adjacent to Y then $CE(S)$ is infinite. A $X - Y$ separator K with $S \cap K = \emptyset$ and $excess(K) = CE(S)$ is called a witness of S (w.r.t. X, Y if not clear from the context).

Lemma 2 Let G be a $X - Y$ -normalized graph and let $S \subseteq N(X)$ and assume that S is not adjacent to Y . There is exactly one important witness $K(S)$ of S .

Proof. Let G' be the graph obtained from G by splitting each $v \in S$ into $n + 1$ copies. It is not hard to see that K' is a witness of S in G if and only if K' is the smallest separator of G' . Furthermore, K' , disjoint with S , is an important $X - Y$ separator of G if and only if K' is an important separator of G' . Combining

the above two statements, we conclude that K' is an important witness of S in G if and only if K' is the smallest important separator of G' . According to Corollary 1, there is exactly one such K' . ■

Remark 1. If $S = \{v\}$, we write $C(v)$ and $K(v)$ instead of $C(\{v\})$ and $K(\{v\})$, respectively. Also, from now on, we will refer to $K(S)$ without special reference to Lemma 2.

Lemma 3 *Let G be a $X - Y$ -normalized graph and let $S \subseteq N(X)$ and assume that S is not adjacent to Y . Let K_1 be an important $X - Y$ separator of G disjoint with S and let $K(S)$ be an important witness of S . Then $K_1 \geq K(S)$.*

Proof. Let G' be the graph as in the first paragraph of the proof of Lemma 2. Since $K(S)$ is the only smallest important $X - Y$ separator of G' , it follows from Corollary 1 that $K' \geq K(S)$ in G' . It is not hard to observe that the same relationship is preserved in G . ■

Lemma 4 *Let G be a $X - Y$ -normalized graph and let $S \subseteq N(X)$ and assume that S is not adjacent to Y . Then there is $S' \subseteq S$ such that $|S'| \leq CE(S)$ and $K(S') = K(S)$.*

Proof. The proof is by induction on $CE(S)$. Assume first that $CE(S) = 1$ and pick an arbitrary vertex $v \in CE(S)$. We claim that $K(S) = K(v)$. Indeed, according to Lemma 3 applied to $\{v\}$, $K(S) \geq K(v)$. Then, according to Proposition 4, $K(S)$ is an $X - Y$ separator of $Pr(G, K(v), X, Y)$ and $Pr(G, K(v), X, Y)$ is normalized. It follows that if $K(S) \neq K(v)$ then $CE(S) = |K(S)| > |K(v)| \geq |N(X)| + 1$, a contradiction. Thus the statement holds in the considered case.

The above reasoning also applies to the case where there is $v \in S$ such that $CE(v) = CE(S)$. Assume this is not the case. Then we can specify a maximal $S^* \subseteq S$ such that $CE(S^*) < CE(S)$. By the induction assumption there is $S'' \subseteq S^*$, $|S''| \leq CE(S^*)$ such that $K(S'') = K(S^*)$. Pick an arbitrary $v \in S \setminus S^*$. We claim that $K(S) = K(S'' \cup \{v\})$. To prove the claim, observe first that $K(S'' \cup \{v\}) = K(S^* \cup \{v\})$. Indeed, according to Lemma 3, $K(S'' \cup \{v\}) \geq K(S'') = K(S^*)$. It follows that $S^* \cup \{v\} \subseteq N(X) \setminus K(S'' \cup \{v\})$. Another application of Lemma 3 shows that $K(S'' \cup \{v\}) \geq K(S^* \cup \{v\})$. On the other hand, $S'' \cup \{v\} \subseteq S^* \cup \{v\}$ and hence, yet another application of Lemma 3 implies $K(S^* \cup \{v\}) \geq K(S'' \cup \{v\})$, yielding the desired equality. Now, observe that $K(S) = K(S^* \cup \{v\})$. Indeed, by Lemma 3, $K(S) \geq K(S^* \cup \{v\})$. On the other hand, due to the minimality of S^* , $K(S) \not\geq K(S^* \cup \{v\})$. The claim now follows. ■

4 Upper bound on the number of important separators and the MWC problem

Let G be an $X - Y$ normalized graph (S_1, \dots, S_r) be a sequence of disjoint non-empty subsets of vertices of G and K is an $X - Y$ separator. We say that K is a *compound witness* of the *attribute* (S_1, \dots, S_r) (w.r.t. X and Y in G if clarification is needed) as follows. Assume first that $r = 1$. Then $S_1 \subseteq N(X)$ and $K = K(S_1)$. Otherwise, $S_2 \cup \dots \cup S_r$ is disjoint with $N(X)$ and K is a compound witness of (S_2, \dots, S_r) w.r.t. X, Y in $Pr(G, X, Y, K(S_1))$. We call $|S_1| + \dots + |S_r|$ the *rank* of K . The following corollary immediately follows from inductive application of Lemma 2.

Corollary 2 *Each sequence (S_1, \dots, S_r) is the attribute of at most one compound witness. (Some sequences may correspond to no compound witness, for example, due to being non well-formed attributes.)*

Theorem 1 *Let G be a $X - Y$ normalized graph and (S_1, \dots, S_r) be a sequence of disjoint non-empty sets of vertices. Then the existence of a compound witness with attribute (S_1, \dots, S_r) can be tested in $O(n^3)$.*

Proof. Consider the following algorithm. First, compute the unique smallest important separator K_0 of $G_0 = G$. Then obtain graph G_1 by introducing extra copies of vertices of S_1 in $Pr(G_0, X, Y, K)$ and compute the smallest important separator K_1 . Then obtain graph G_2 from $Pr(G_1, X, Y, K_1)$ by introducing extra copies of vertices of S_2 and so on until K_r is eventually returned. The algorithm can also return 'NO' if some X or some intermediate K_i is adjacent to Y or if some S_i is not a subset of K_{i-1} . The correctness of this algorithm follows from definition of the attribute.

The runtime $O(rn^3)$ immediately follows from Lemma 1. However, using an amortisation argument we can show that in fact $O(n^3)$ is enough. Denote $|K_i|$ by z_i and assume w.l.o.g. that K_r is successfully computed (otherwise we can consider computation until some $K_{r'}$ for $r' < r$. By Proposition 4, $z_0 < \dots < z_r$. Now, consider graph G_1 . It is not hard to see that the z_0 internally vertex disjoint $X - Y$ paths of G_0 (found during the run of network flow algorithm) are naturally transformed into z_0 internally vertex disjoint $X - Y$ paths of G_1 . These paths provide initial flow of size z_0 and hence only $(z_1 - z_0) + 1$ additional iterations of the Ford-Fulkerson algorithm will be needed for the next iteration of the algorithm of Lemma 1 to produce the largest set of internally vertex disjoint $X - Y$ path of G_1 . Applying this argument inductively, it is not hard to observe that the resulting algorithm takes $O(n)$ iterations of Ford-Fulkerson algorithm. Each of these iterations takes $O(n^2)$. In addition there are at most n iterations of computing the smallest important separator, each requiring $O(n^2)$ time according to Lemma 1. Finally, the algorithm also creates a Pr -graph at most n times, $O(n^2)$ per creation is clearly enough. Consequently, the overall runtime is $O(n^3)$. ■

Theorem 2 *Let G be a $X - Y$ -normalized graph and let $K \neq N(X)$ an important $X - Y$ separator. Then K is a compound witness of rank at most $excess(K)$.*

Proof. By induction on $excess(K)$. Assume first that $excess(K) = 1$ and let $v \in N(X) \setminus K$. Then $K = K(v)$, as shown in the first paragraph of proof of Lemma 4. In other words, in the considered case, K is a compound witness with attribute $(\{v\})$.

Assume now that $excess(K) > 1$. Denote $N(X) \setminus K$ by S . According to Lemma 3, $K \geq K(S)$. Furthermore, according to Lemma 4, there is $S_1 \subseteq S$ with $|S_1| \leq CE(S)$ such that $K(S) = K(S_1)$. If $K = K(S)$ then (S_1) is the desired attribute. Otherwise, denote $P(G, X, Y, K(S))$ by G_1 . According to Proposition 4, G_1 is normalized and K is an important $X - Y$ separator of G_1 . Furthermore, $excess_{G_1, X, Y} K = excess_{G, X, Y} K - CE(S_1) < excess_{G, X, Y} K$. By the induction assumption, K is a compound witness w.r.t. X, Y in G_1 of rank at most $excess_{G_1, X, Y} K$. Let (S_2, \dots, S_r) be the corresponding attribute. We claim that K is the compound witness of (S_1, \dots, S_r) w.r.t. X, Y in G . Indeed, $|S_1| + \sum_{i=2}^r |S_i| \leq CE(S_1) + excess_{G_1, X, Y} K = excess_{G, X, Y} K$, the inequality is obtained by definition of S_1 and the induction assumption, the equality is obtained by definition of G_1 .

It remains to show that S_2, \dots, S_r are disjoint with $N(X)$. First of all, note that K is disjoint with S_1 . Furthermore, inductively applying the definition of a compound witness, it is not hard to see that K is disjoint with S_2, \dots, S_r . Since each of S_2, \dots, S_r are subsets of vertices of $Pr(G, X, Y, K(S))$, they are all disjoint with S . It follows that if some S_i is not disjoint with $N(X)$, it is in fact not disjoint with $N(X) \setminus S$. Let $v \in (N(X) \setminus S) \cap S_i$. It follows that $v \notin K$ in contradiction to $N(X) \setminus K = S$. ■

Theorem 3 *Let G be a graph and let X and Y be two non-intersecting subsets of $V(G)$. Let $k > 0$ be an integer. Then there are at most $\sum_{i=0}^k \binom{n}{i}$ important $X - Y$ separators of excess at most k . Moreover, they can be generated by considering all subsets of at most k vertices of G with an $O(n^3)$ time spent per subset.*

Proof. First of all we show that we can assume that G is an $X - Y$ normalized graph. Indeed, assume that G is not such graph and let K^* be the only smallest important separator existing according to Corollary 1. Let K' be an arbitrary important separator. According to Corollary 1, $K' \geq K^*$. It follows from Proposition 4 that the set of important $X - Y$ separators of G is the same as the set of important $X - Y$ separators of $Pr(G, X, Y, K^*)$ and that $Pr(G, X, Y, K^*)$ is normalized. This shows the validity of assumption that G is

an $X - Y$ normalized graph. The $\binom{n}{0}$ in the claimed bound stands for the unique smallest important $X - Y$ separator, $N(X)$ in our case. We are now going to show that the number of the rest of important $X - Y$ separators is at most $\sum_{i=1}^k \binom{n}{i}$.

Let us say that a set S *corresponds* to an attribute (S_1, \dots, S_r) (and vice versa the attribute corresponds to the set) if $\bigcup_{i=1}^r S_i = S$. We show that each subset S of $V(G)$ corresponds to at most one well-formed attribute (S_1, \dots, S_r) of a compound witness. The proof is by induction. The empty set does not correspond to any well-formed attribute. Assume that $|S| = 1$. If S is disjoint with $N(X)$ then again S does not correspond to any well-formed attribute. Otherwise, $S \subseteq N(X)$ and the only attribute S can correspond to is (S) . Assume now that $|S| > 1$. If S is disjoint with $N(X)$ then once again S does not correspond to any well-formed attribute. Otherwise, let (S_1, \dots, S_r) be an attribute corresponding to S . Observe that $S_1 = S \cap N(X)$. Furthermore, by the induction assumption, (S_2, \dots, S_r) is the unique attribute corresponding to $S \setminus S_1$. Taking into account the uniqueness of S_1 , the uniqueness of (S_1, \dots, S_r) follows.

The correspondence established above tells us that there are at most $\sum_{i=1}^k \binom{n}{i}$ well-formed attributes of rank at most k . Since according to Corollary 2, each (S_1, \dots, S_r) is the attribute of at most one compound witness w.r.t. X and Y , the number of compound witnesses of rank at most k is also bounded by $\sum_{i=1}^k \binom{n}{i}$. Theorem 2 implies the same bound on the number of important $X - Y$ separators different from $N(X)$ and having excess at most k . Finally, the runtime upper bound follows from Theorem 1. ■

With Theorem 3 in mind we are ready to compute the runtime of solving MWC problem. Let (G, T) be an instance of the multiway cut problem where G is a graph and T is the set of terminals to be separated. Let $t \in T$. We call a $t - T \setminus t$ separator of G an *isolating cut* of t (w.r.t. (G, T) if the context is not clear). The following lemma has is a reformulation of Lemma 3.6. of [16].

Lemma 5 *For any $t \in T$ there is an optimal solution of (G, T) containing an important isolating cut of t .*

Theorem 4 *Let (G, T) be an instance of the multiway cut problem. For $t \in T$, let $m(t)$ be the size of the smallest isolating cut of t . Let $m = \max_{t \in T} m(t)$ and s be an integer. Then there is $O(sn^{s+3} + |T|n^3)$ algorithm that checks whether (G, T) has a solution of size at most $(m + s)$.*

Proof. For each terminal of T compute the respective smallest important isolating cut. According to Lemma 1, this can be done in $O(n^3)$ per terminal, so the overall time spent in $O(|T|n^3)$. Let t be the terminal whose respective smallest important isolating cut is of size m . If $k = 0$ then, according to Lemma 5 and Corollary 1, either this isolating cut is the solution or there is no solution.

If $k > 0$, the algorithm generates all possible important isolating cuts K of t of excess at most k . For each such K , it solves the instance $(G \setminus K, T \setminus \{t\}, m + k - |K|)$ and returns 'YES' if and only if at least one such residual instance has a solution. The correctness of this approach follows from Lemma 5. Furthermore, since $|K| \geq m$, $m + k - |K| \leq k$.

According to [4], each residual instance can be solved in time $O(n^3(k - i)4^{k-i})$, where i is the excess of K . According to Theorem 3, for each $i \leq k$ there are at most $\sum_{j=0}^i \binom{n}{j}$ important isolating cuts of t of excess i . Moreover, they can be enumerated by spending $O(n^3)$ for each of them. The proposed approach requires to spend additional time $O(n^3(k - i)4^{k-i})$ per isolating cut of t . The overall time spent per an isolating cut of t is thus $O(n^3(k - i)4^{k-i} + n^3) \subseteq O(n^3k4^{k-i})$. Taking into account that $\sum_{j=0}^i \binom{n}{j} \leq c \binom{n}{i} n^i$ for some constant c the resulting runtime is $O(n^3k \sum_{i=0}^k \binom{k}{i} n^i 4^{k-i}) = O(n^3k(n + 4)^k)$. The desired runtime can be obtained by taking into account that $(n + 4)^k$ and n^k are asymptotically the same. ■

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